

Entanglement entropy for pure gauge theories in 1+1 dimensions using the lattice regularization

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ABSTRACT: We study the entanglement entropy (EE) for pure gauge theories in 1+1 dimensions with the lattice regularization. Using the definition of the EE for lattice gauge theories proposed in a previous paper [1], we calculate the EE for arbitrary pure as well as mixed states in terms of eigenstates of the transfer matrix in 1+1 dimensional lattice gauge theory. We find that the EE of an arbitrary pure state does not depend on the lattice spacing, thus giving the EE in the continuum limit, and show that the EE for an arbitrary pure state is independent of the real (Minkowski) time evolution. We also explicitly demonstrate the dependence of EE on the gauge fixing at the boundaries between two subspaces, which was pointed out for general cases in the paper [1]. In addition, we calculate the EE at zero as well as finite temperature by the replica method, and show that our result in the continuum limit corresponds to the result obtained before in the continuum theory, with a specific value of the counter term, which is otherwise arbitrary in the continuum calculation. We confirm the gauge dependence of the EE also for the replica method.

KEYWORDS: Entanglement entropy, pure gauge theories, lattice regularization, transfer matrix, replica method

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1 Introduction

The entanglement entropy (EE), which tells us quantum properties of a given state, plays important roles in many fields of physics including quantum field theories [2, 3], the string theory [4–11], condensed matter physics [12–17] and the physics of the black hole [18–22]. For example, the EE is thought to be a useful tool to investigate confinement/deconfinement phase transition in gauge theories, in the contexts of the Gauge/Gravity correspondence [23–25] as well as a purely field theoretical approach [26–28].

To calculate the EE of a region V , one needs to express the whole Hilbert space as a tensor product of the Hilbert space of V and that of the complement of V (denoted as \bar{V}). Due to the local gauge invariance in gauge theories, however, the gauge invariant Hilbert space, which is characterized by gauge invariant operators such as Wilson loops, cannot be decomposed into a tensor product of the gauge invariant subspaces of V and \bar{V} . Because of this problem, there is no unique way to define the EE in gauge theories [29–33].

In the previous paper [1], one of the present authors (S.A.) with collaborators proposed a definition of the entanglement entropy in lattice gauge theories. They simply extended the gauge invariant Hilbert space to the whole Hilbert space of the link variables, which is then decomposed into a tensor product of the Hilbert spaces of the link variables in the region V and those in the region \bar{V} . Using this decomposition, the EE can be defined for an arbitrary

subset of links, and the definition can be applied to both abelian and non-abelian gauge theories. They also discussed the issue of gauge invariance and pointed out the EE depends on the gauge fixing at the boundaries between V and \bar{V} . They applied their definition of the EE to Z_N lattice gauge theories and investigated several properties including the gauge dependence and the EE of the topological states in arbitrary dimensions. Similar proposals have been also made in refs. [34–36].

In this paper, using the definition of the EE in ref. [1], we further study the EE for gauge theories based on the lattice regularization, which is nonperturbative and gauge invariant. As a simple but non-trivial model, we consider pure gauge theories in 1+1 dimensions with the lattice regularization, and show that the EE can be calculated analytically using the eigenstate of the transfer matrix in 1+1 dimensional lattice gauge theories. We find that the EE does not depend on both lattice spacing and lattice size, so that the EE for an arbitrary pure state in the continuum limit can be automatically obtained. We also investigate the issue of the gauge dependence explicitly calculating the EE with various choices of the gauge fixing, and demonstrate that the EE depends on the gauge fixing if gauge transformations at boundaries between two subspaces are employed.

The present paper is organized as follows. In section 2 we give a brief summary of lattice gauge theories in 1+1 dimensions such as the character expansion and the transfer matrix. In section 3, we calculate the EE for arbitrary pure as well as mixed states in the lattice gauge theories in 1+1 dimensions, using gauge invariant states in the Hilbert space (called the operator method), where any states can be expanded in terms of eigenstates of the transfer matrix. We show that the EE of an arbitrary pure state does not depend on the lattice spacing, so that the continuum limit is automatically realized. In addition, the EE for an arbitrary pure state is found to be time independent. We also demonstrate that the EE depends on the gauge fixing at the boundaries, as pointed out in the previous paper [1], by explicitly calculating the EE with various gauge fixings. In section 4, we conclude the paper. In appendix A, we calculate the EE for pure gauge theories in 1+1 dimensions on the lattice, using the replica method, at zero as well as finite temperatures. We show that the EE in the continuum limit agrees with the EE in the continuum theory. Furthermore, the EE obtained from lattice gauge theories determines a value of the counter term, which cannot be fixed in the continuum calculation, showing one of the advantages of the lattice regularization. We finally confirm that the EE depends on the gauge fixing at the boundaries also in this case.

2 Lattice gauge theories in 1+1 dimensions

In this section, we collect several formula for lattice gauge theories in 1+1 dimensions, which will be used for the latter sections.

2.1 Basic formula

We denote $U(R)$ the (unitary) irreducible representation R of $U \in G$ for a group G . The plaquette action for gauge theories with the gauge group G is given by

$$S_p = \beta \sum_{n \in \mathbb{Z}^2} \text{tr} \left(U_{P,n}(F) + U_{P,n}^\dagger(F) - \mathbf{2} \right), \quad (2.1)$$

where the plaquette $U_{P,n}$ is defined as

$$U_{P,n} = U_{n,1} U_{n+\hat{1},2} U_{n+\hat{2},1}^\dagger U_{n,2}^\dagger, \quad (2.2)$$

F represents the fundamental representation (for example, $U(F)$ is an $N \times N$ unitary matrix for $G = \text{SU}(N)$ or $\text{U}(N)$), and $\hat{\mu}$ is the unit vector in the μ direction ($\mu = 1, 2$). The inverse gauge coupling β is related to the gauge coupling g

$$\beta = \frac{1}{g^2 a^2}, \quad (2.3)$$

where g has the mass dimension one and a is the lattice spacing.

The character expansion of each plaquette is given by

$$\exp \left[\beta \chi_F \left(U_{P,n} + U_{P,n}^\dagger - \mathbf{2} \right) \right] = \sum_R d_R \lambda_R(\beta) \chi_R(U_{P,n}), \quad (2.4)$$

where

$$\chi_R(U) = \text{tr} U(R), \quad d_R = \chi_R(\mathbf{1}), \quad (2.5)$$

$$\lambda_R(\beta) = \frac{1}{d_R} \int dU \chi_R(U) \exp \left[\beta \chi_F \left(U + U^\dagger - \mathbf{2} \right) \right]. \quad (2.6)$$

This definition leads to

$$0 \leq \lambda_R(\beta) \leq 1, \quad \lambda_R(\beta) = 1 \Leftrightarrow \beta = \infty. \quad (2.7)$$

For example, for the spin j representation of $G = \text{SU}(2)$, we have

$$\lambda_j(\beta) = \frac{e^{-4\beta} I_{2j+1}(4\beta)}{4\beta} \quad (2.8)$$

for half integer j , where I_n is the modified Bessel function. Formula of $\lambda_R(\beta)$ for other gauge groups can be found in ref. [37].

We give two important formula,

$$\int d\Omega \chi_R(A\Omega) \chi_{R'}(\Omega^\dagger B) = \frac{1}{d_R} \delta_{RR'} \chi_R(AB), \quad (2.9)$$

$$\int d\Omega \chi_R(A\Omega B\Omega^\dagger) = \frac{1}{d_R} \chi_R(A) \chi_R(B), \quad (2.10)$$

which follow from

$$\int d\Omega \Omega_{ab}(R) \Omega_{cd}^\dagger(R') = \frac{1}{d_R} \delta_{RR'} \delta_{ad} \delta_{bc}. \quad (2.11)$$

2.2 Transfer matrix, eigenvalues and eigenstates

The transfer matrix \hat{T} for the plaquette action (2.1) on 1+1 dimensional lattice is given in ref. [1] as

$$T(U, V) \equiv \langle U | \hat{T} | V \rangle = \prod_{x=0}^{L-1} \exp \left\{ \beta \text{tr} \left[\left(U_x V_x^\dagger + V_x U_x^\dagger - \mathbf{2} \right) \right] \right\} \quad (2.12)$$

where U_x, V_x are spatial link variables and L is the number of links in 1-dimensional lattice. Using the character expansion, we write

$$T(U, V) = \prod_{x=0}^{L-1} \sum_R d_R \lambda_R(\beta) \chi_R(U_x V_x^\dagger). \quad (2.13)$$

An eigenfunction of the transfer matrix is easily obtained with the periodic boundary condition (PBC) as

$$R(U) \equiv \langle U | R \rangle = \chi_R(U), \quad U \equiv \prod_{x=0}^{L-1} U_x, \quad (2.14)$$

which satisfies

$$\begin{aligned} \langle U | \hat{T} | R \rangle &= \int \mathcal{D}V T(U, V) R(V) = \prod_{x=0}^{L-1} \int dV_x \sum_{R_x} d_{R_x} \lambda_{R_x}(\beta) \chi_{R_x}(U_x V_x^\dagger) \chi_R(V) \\ &= \lambda_R^L(\beta) \chi_R \left(\prod_{x=0}^{L-1} U_x \right) = \lambda_R^L(\beta) \langle U | R \rangle, \end{aligned} \quad (2.15)$$

so that the eigenvalue is $\lambda_R^L(\beta)$, and has the correct normalization as

$$\langle R' | R \rangle = \int \prod_{x=0}^{L-1} dU_x \chi_{R'}(U^\dagger) \chi_R(U) = \delta_{R'R}. \quad (2.16)$$

To understand the nature of the state $|R\rangle$, let us consider $G = U(1)$ case, where $R = n$ is an integer. The positive (negative) n represents how many times the Wilson line warps around the circle, the 1-dimensional space with the PBC, in the positive (negative) direction.

3 Direct calculations of EE in lattice gauge theories

In this section, we directly calculate the EE from the operator method of the 1+1 dimensional lattice gauge theory.

3.1 Density matrix and entanglement entropy

We first consider the density matrix for an eigenstate $|R\rangle$ as

$$\rho(R) = |R\rangle\langle R|, \quad (3.1)$$

where we can take not only the ground state $|0\rangle$ but also excited states $|R \neq 0\rangle$. We here assume the PBC in space (*i.e.* 1-dimensional torus).

We take regions A and \bar{A} as

$$A = \bigcup_{i=1}^{\ell} A_i, \quad \bar{A} = \bigcup_{i=1}^{\ell} B_i, \quad (3.2)$$

where $A_i \cap A_j = \emptyset$ and $B_i \cap B_j = \emptyset$ for $i \neq j$, and $A \cap \bar{A} = \emptyset$ by definition.

The reduced density matrix is defined as

$$\rho_A(R) = \text{tr}_{\mathcal{H}_{\bar{A}}} |R\rangle\langle R|, \quad (3.3)$$

which is explicitly given by

$$\begin{aligned} {}_A\langle U | \rho_A(R) | V \rangle_A &= \prod_{i=1}^{\ell} \int dW_i \chi_R(U_1 W_1 \cdots U_{\ell} W_{\ell}) \chi_R(W_{\ell}^{\dagger} V_{\ell}^{\dagger} \cdots W_1^{\dagger} V_1^{\dagger}) \\ &= d_R^{-\ell} \prod_{i=1}^{\ell} \chi_R(U_i V_i^{\dagger}), \end{aligned} \quad (3.4)$$

where U_i or V_i is the ordered product of links in A_i , while W_i is that in B_i . Using this, we can show

$${}_A\langle U | \rho_A(R)^2 | V \rangle_A = d_R^{-3\ell} \prod_{i=1}^{\ell} \chi_R(U_i V_i^{\dagger}) = d_R^{-2\ell} {}_A\langle U | \rho_A(R) | V \rangle_A, \quad (3.5)$$

which means

$$\rho_A^2(R) = d_R^{-2\ell} \rho_A(R), \quad \rho_A^n(R) = d_R^{-2(n-1)\ell} \rho_A(R), \quad (3.6)$$

so that

$$\text{tr}_{\mathcal{H}_A} \rho_A^n(R) = d_R^{-2(n-1)\ell} \text{tr}_{\mathcal{H}_A} \rho_A(R) = d_R^{-2(n-1)\ell}. \quad (3.7)$$

Therefore, the EE is obtained as

$$S_R(A) = -\text{tr}_{\mathcal{H}_A} [\rho_A \log \rho_A] = \lim_{n \rightarrow 1} \frac{1}{1-n} \log \{\text{tr}_{\mathcal{H}_A} \rho_A^n(R)\} = 2\ell \log d_R, \quad (3.8)$$

showing that the EE is zero for the vacuum state ($R=0$) as $d_0=1$. Remarkably, the EE depends on neither the lattice spacing a nor the number of lattice points L , so that the EE in the continuum limit is nothing but eq. (3.8). Furthermore, since the EE in eq. (3.8) only depends on ℓ , the number of disjoint components of A , the state $|R\rangle$ is regarded as

the topological state, where the EE is insensitive to both position and size of A . Since the EE does not depend on the size of A , the strong subadditivity is trivially satisfied. Note that the appearance of $\log d_R$ contributions to the EE in non-abelian gauge theories has been pointed out in refs. [1, 36].

We next consider a state consisting of a linear combination of eigenstates as

$$|\{c_R\}\rangle \equiv \sum_R c_R |R\rangle, \quad \sum_R |c_R|^2 = 1, \quad (3.9)$$

whose density matrix is written as

$$\rho(\{c_R\}) = \sum_{R,R'} c_R \bar{c}_{R'} |R\rangle \langle R'|. \quad (3.10)$$

The reduced density matrix is thus given by

$$\rho_A(\{c_R\}) = \sum_R |c_R|^2 \rho_A(R), \quad (3.11)$$

so that

$$\rho_A^n(\{c_R\}) = \sum_R |c_R|^{2n} d_R^{-2(n-1)\ell} \rho_A(R). \quad (3.12)$$

We thus obtain

$$\text{tr}_{\mathcal{H}_A} \rho_A^n(\{c_R\}) = \sum_R p_R^n d_R^{-2(n-1)\ell}, \quad p_R \equiv |c_R|^2, \quad (3.13)$$

$$S_{\{c_R\}}(A) = 2\ell \sum_R p_R \log d_R - \sum_R p_R \log p_R. \quad (3.14)$$

Again the EE in eq. (3.14) is considered to be the result in the continuum limit, since it does not depend on the lattice spacing a . In ref. [36], these two contributions to the EE in non-abelian gauge theories are called classical. It was also argued that these two contributions cannot be extracted in dilution or distillation experiments which involve only Local Operations and Classical Communication (LOCC). See ref. [36] for more details.

Let us consider the real-time dependence of the state $|R\rangle$, controlled by the Schrödinger equation as

$$\hat{H}|R, t\rangle = \frac{\partial}{\partial t}|R, t\rangle, \quad |R, 0\rangle = |R\rangle, \quad \hat{H} = -\frac{1}{a} \log \hat{T}, \quad (3.15)$$

which can be solved as

$$|R, t\rangle = e^{iE_R t} |R\rangle, \quad E_R = -\frac{L}{a} \log \lambda_R(\beta). \quad (3.16)$$

Therefore, the time dependence for the state $|\{c_R\}\rangle$ is given by $|\{c_R(t)\}\rangle$ with

$$c_R(t) = c_R e^{iE_R t}, \quad (3.17)$$

which however gives

$$p_R(t) = |c_R(t)|^2 = |c_R|^2 = p_R. \quad (3.18)$$

This means that the EE for this state is independent under the real-time evolution described by the hamiltonian not only in the continuum limit but also on the lattice. Since the state here is not a thermal state, this time independence of the EE is nontrivial and is a special feature of gauge theories in $1+1$ dimensions.

Finally we consider the density matrix for the general mixed states given by

$$\rho(\{p_R\}) \equiv \sum_R p_R |R\rangle \langle R|. \quad (3.19)$$

For example, for a thermal state at temperature T_B , we have

$$p_R = \frac{e^{-E_R/T_B}}{\sum_R e^{-E_R/T_B}} = \frac{\lambda_R^{L/(aT_B)}}{\sum_R \lambda_R^{L/(aT_B)}}, \quad \sum_R p_R = 1, \quad p_R \geq 0. \quad (3.20)$$

The reduced density matrix for this state becomes

$$\rho_A(\{p_R\}) = \sum_R p_R \rho_A(R), \quad (3.21)$$

and therefore we have

$$\rho_A^n(\{p_R\}) = \sum_R p_R^n d_R^{-2(n-1)\ell} \rho_A(R), \quad (3.22)$$

which leads to the same EE as before:

$$S_{\{p_R\}}(A) = 2\ell \sum_R p_R \log d_R - \sum_R p_R \log p_R, \quad (3.23)$$

where the second term is equal to the von Neumann entropy originated from the mixed state as

$$S_{\text{mix}} = -\text{tr}[\rho(\{p_R\}) \log \rho(\{p_R\})] = -\sum_R p_R \log p_R. \quad (3.24)$$

For the thermal state, we have

$$S_{\{p_R\}}(A) = \log \sum_R \lambda_R^{L/(aT_B)}(\beta) + \frac{\sum_R \lambda_R^{L/(aT_B)}(\beta) \left(2\ell \log d_R - \frac{L}{aT_B} \log \lambda_R(\beta) \right)}{\sum_R \lambda_R^{L/(aT_B)}(\beta)}, \quad (3.25)$$

which depends on the lattice spacing a , the temperature T_B and the number of lattice points L . As shown in the appendix, this result agrees with the one obtained from the replica method. We will also see that this EE in the continuum limit reproduces the previous result obtained in the calculation of the continuum theory [38].

3.2 Gauge fixing and gauge invariance

As discussed in ref. [1], the EE is gauge invariant in the sense that it does not depend on the gauge fixing as long as no gauge fixing is employed at the boundary points while gauge transformations including those at boundary points can change the value of the EE. We explicitly demonstrate these properties mainly for the state $|R\rangle$ below.

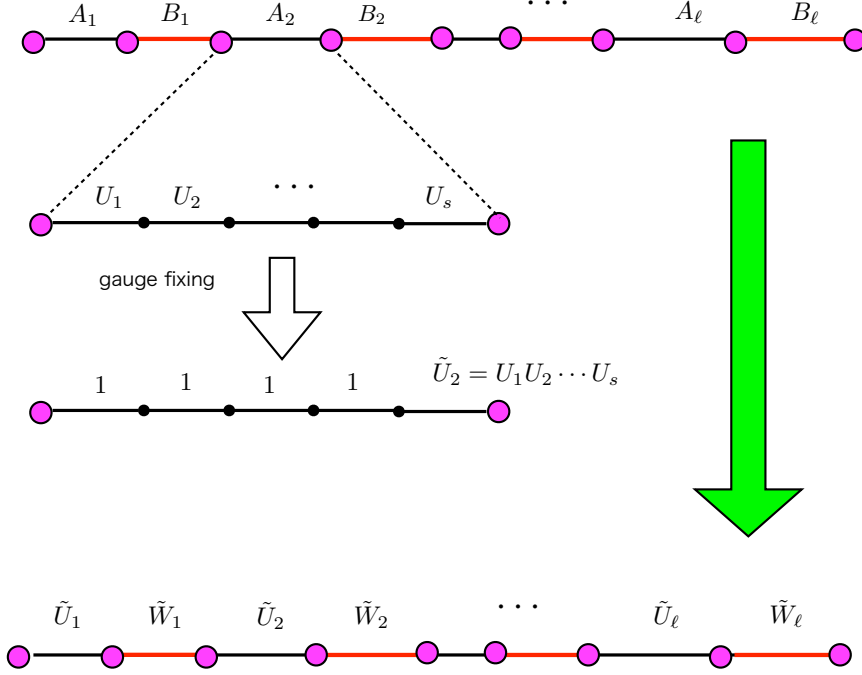


Figure 1. The gauge fixing without gauge transformation at boundaries.

We can set all link variables except one to an unit matrix in each A_i or B_i , using gauge transformations at lattice points inside each region without boundary points so that $A_i \rightarrow \tilde{U}_i$ and $B_i \rightarrow \tilde{W}_i$ as shown in fig. 1. The corresponding reduced density matrix for the state $|R\rangle$ can be calculated as

$$\begin{aligned}
 {}_A\langle\tilde{U}|\rho_A(R)|\tilde{V}\rangle_A &= \prod_{i=1}^{\ell} \int d\tilde{W}_i \chi_R(\tilde{U}_1 \tilde{W}_1 \cdots \tilde{U}_\ell \tilde{W}_\ell) \chi_R(\tilde{W}_\ell^\dagger \tilde{V}_\ell^\dagger \cdots \tilde{W}_1^\dagger \tilde{V}_1^\dagger) \\
 &= d_R^{-\ell} \prod_{i=1}^{\ell} \chi_R(\tilde{U}_i \tilde{V}_i^\dagger),
 \end{aligned} \tag{3.26}$$

which gives

$$S_R(A) = 2\ell \log d_R. \tag{3.27}$$

This shows that the EE after the gauge fixing remains the same as eq. (3.8) without gauge fixing.

We next consider the extreme case where gauge transformation at all points including all boundaries are used to fix the gauge. In this case, we can fix all link variables to an unit matrix except one (due to the PBC), which we take \tilde{U}_1 in A_1 . The reduced density matrix is given as

$${}_A\langle\tilde{U}_1|\rho_A(R)|\tilde{V}_1\rangle_A = \chi_R(\tilde{U}_1)\chi_R(\tilde{V}_1^\dagger), \quad (3.28)$$

which leads to $S_R(A) = 0$.

We finally consider more general case where gauge transformations including those at some boundary points are employed. We fix link variables to an unit matrix except a few so that non-trivial link variables are given in the following order

$$\tilde{U}_1, \tilde{W}_1, \tilde{U}_2, \tilde{W}_2, \dots, \tilde{U}_s, \tilde{W}_s, \quad \tilde{U}_i \in A, \tilde{W}_i \in \bar{A}. \quad (3.29)$$

The corresponding reduced density matrix for the state $|R\rangle$ becomes

$$\begin{aligned} {}_A\langle\tilde{U}|\rho_A(R)|\tilde{V}\rangle_A &= \prod_{i=1}^s \int d\tilde{W}_i \chi_R(\tilde{U}_1 \tilde{W}_1 \dots \tilde{U}_s \tilde{W}_s) \chi_R(\tilde{W}_s^\dagger \tilde{V}_s^\dagger \dots \tilde{W}_1^\dagger \tilde{V}_1^\dagger) \\ &= d_R^{-s} \prod_{i=1}^s \chi_R(\tilde{U}_i \tilde{V}_i^\dagger), \end{aligned} \quad (3.30)$$

which gives

$$S_R(A) = 2s \log d_R, \quad s = 1, 2, \dots, \ell. \quad (3.31)$$

Here the order of U and W is important to obtain the above result. For example, the order $\tilde{U}_1, \tilde{W}_1, \tilde{U}_2, \tilde{W}_2$ corresponds to $s = 2$, while $\tilde{U}_1, \tilde{W}_1, \tilde{W}_2, \tilde{U}_2$ to $s = 1$ because of the PBC.

We thus conclude that a possible value of the EE for the state $|R\rangle$ is given as

$$S_R(A) = 2s \log d_R, \quad s = 0, 1, 2, \dots, \ell, \quad (3.32)$$

by some choice of the gauge transformations.

For the general state, it is easy to see that

$$S_{\{p_R\}}(A) = 2s \sum_R p_R \log d_R - \sum_R p_R \log p_R, \quad s = 0, 1, 2, \dots, \ell, \quad (3.33)$$

where $p_R = |c_R|^2$ for the state $|\{c_R\}\rangle$.

4 Conclusion

In this paper, we calculate the EE for the 1+1 dimensional pure gauge theories using the lattice regularization with the operator method, and obtain

$$S(A) = \sum_R p_R (2\ell \log d_R - \log p_R), \quad p_R = |c_R|^2 \quad (4.1)$$

for the state

$$|\{c_R\}\rangle = \sum_R c_R |R\rangle, \quad (4.2)$$

where $|R\rangle$ is the eigenstate of the transfer matrix and R specifies the irreducible representation of the gauge group G . This result can be regarded as the continuum one as it does not depend on the lattice spacing a . A similar result is also obtained for the mixed states including the thermal state.

We explicitly confirm that the above EE can be reduced by the gauge transformation as

$$S(A) = \sum_R p_R (2s \log d_R - \log p_R), \quad s = 0, 1, \dots, \ell, \quad (4.3)$$

as pointed out in ref. [1].

In appendix A, we calculate the same quantities using the replica method. We will confirm that the results obtained by the replica method reproduce the EE for the vacuum state $|0\rangle$ as well as the thermal state in the main text. We also confirm that the EE for the thermal state in the continuum limit reproduce the known continuum result [38]. In addition, the value of the counter term can be fixed by the lattice calculation, contrary to the continuum treatment, which leave this term arbitrary [38]. A similar gauge dependence of the EE will be also demonstrated.

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A Entanglement Entropy from the replica method

In this appendix, we calculate the EE for 1+1 dimensional lattice gauge theories using the replica method.

A.1 Observables of lattice gauge theories in 1+1 dimensions

This subsection includes some useful formula of lattice gauge theories, which will be used in this appendix.

Let us consider the quantity defined by

$$K_C(\Gamma) = \int \prod_{\ell \in C_{\text{in}}} dU_\ell e^{S_p(C)}, \quad \Gamma = \prod_{\ell \in C} U_\ell \quad (A.1)$$

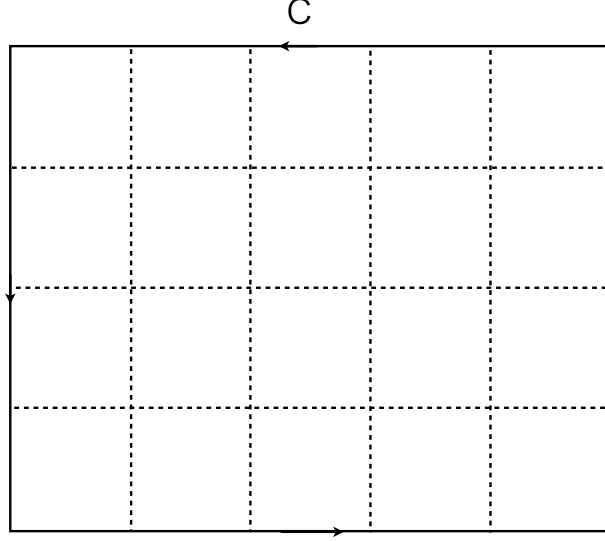


Figure 2. An example of a closed loop C . The dashed links belong to C_{in} , while Γ is the ordered product of links on C . The action $S_p(C)$ consists of plaquettes inside C , ones made of dashed links only and the others made of both dashed and solid links. $\#C$, which is the number of the plaquettes, is 20 in this loop C .

for the closed loop C (for example, see fig. 2), and C_{in} represents the links inside C (dashed links in fig. 2) without links on C . Here $\ell = (n, \mu)$ represents a link between n and $n + \hat{\mu}$ and $S_p(C)$ is the action consisting of the plaquettes inside the loop C . Using the formula in eqs. (2.9) and (2.10), we obtain

$$K_C(\Gamma) = \sum_R d_R \lambda_R^{\#C}(\beta) \chi_R(\Gamma), \quad (\text{A.2})$$

where $\#C$ is the number of plaquettes inside C .

Thus, the partition function with the PBC in both directions is given by

$$Z_{\text{PBC}} = \sum_R \lambda_R^{LT}(\beta), \quad (\text{A.3})$$

where LT is the total number of space-time plaquettes.

We next consider the expectation value of a $L_0 \times T_0$ Wilson loop for the irreducible representation $R (\neq 0)$ with the PBC. Since the product of two irreducible representations is decomposed as

$$R \otimes R_a = \bigoplus_b N_{R,R_a}^{R_b} R_b, \quad (\text{A.4})$$

which means each irreducible representation R_b appears $N_{R,R_a}^{R_b}$ times in the product of

$R \otimes R_a$, we have

$$\langle \chi_R(U_{L_0 \times T_0}) \rangle = \sum_{R_a, R_b} \frac{d_{R_a} d_{R_b}}{Z_{\text{PBC}}} N_{R, R_a}^{R_b} \lambda_{R_b}^{L_0 T_0}(\beta) \lambda_{R_a}^{LT - L_0 T_0}(\beta). \quad (\text{A.5})$$

Assuming that $L_0 T_0 \ll LT$ and taking the large LT limit, the term with $R_a = 0$ dominates in the above, so that $R_b = R$ and $N_{R,0}^R = 1$, which leads to

$$\langle \chi_R(U_{L_0 \times T_0}) \rangle \simeq d_R \left(\frac{\lambda_R(\beta)}{\lambda_0(\beta)} \right)^{L_0 T_0}, \quad (\text{A.6})$$

where we use the fact that $\lambda_0(\beta) > \lambda_R(\beta)$ ($R \neq 0$) for all $\beta < \infty$. Thus the static quark potential for the representation R is given by

$$V_R(L_0)a = - \lim_{T_0 \rightarrow \infty} \frac{1}{T_0} \log \langle \chi_R(U_{L_0 \times T_0}) \rangle = -L_0 \log \left(\frac{\lambda_R(\beta)}{\lambda_0(\beta)} \right), \quad (\text{A.7})$$

which increases linearly in L_0 , showing the confinement. The string tension in the physical unit is obtained as

$$\sigma_R a^2 = - \log \left(\frac{\lambda_R(\beta)}{\lambda_0(\beta)} \right). \quad (\text{A.8})$$

In the continuum limit that $\beta \rightarrow \infty$, we have

$$\lim_{\beta \rightarrow \infty} \left(\frac{\lambda_R(\beta)}{\lambda_0(\beta)} \right) = \frac{1 - c(R)\beta^{-1}}{1 - c(0)\beta^{-1}} + O(\beta^{-2}) \simeq 1 - g^2 a^2 C_2(R) \quad (\text{A.9})$$

where $C_2(R) = c(R) - c(0)$. For $\text{SU}(2)$, we have

$$c(j) = (j + 1/4)(j + 3/4), \quad C_2(j) = j(j + 1) \quad (\text{A.10})$$

for $j = 0, 1/2, 1, 3/2, 2, \dots$. Note that $C_2(j)$ is the quadratic Casimir of the spin j representation, and this is true in general that $C_2(R)$ is the quadratic Casimir of the irreducible representation R of the group G , defined by

$$\sum_a T^a(R) T^a(R) = C_2(R) \mathbf{1}, \quad (\text{A.11})$$

where $T^a(R)$ is the irreducible representation R for the generator T^a of the group G . For example, $C_2(n) = n^2$ for the $R = n \in \mathbb{Z}$ representation of $\text{U}(1)$ group, while $C_2(q_1, q_2) = q_1 + q_2 + (q_1^2 + q_1 q_2 + q_2^2)/3$ for the representation (q_1, q_2) of $\text{SU}(3)$ group, where $q_i = (\text{number of boxes in row } i) - (\text{number of boxes in row } (i + 1))$ in the Young tableau of $\text{SU}(N)$ group [37]. Casimir invariants for a few low-lying representations of $\text{SU}(N)$ group can be found in ref. [39] and are given in Table 1, where \overline{n} represents $N - n$ boxes in a column.

Using the formulas, the string tension for the irreducible representation R is given by

$$\sigma_R = C_2(R) g^2 \quad (\text{A.12})$$

in the continuum limit.

Rep.	$(q_1, q_2, \dots, q_{N-1})$	d_R	$C_2(R)$
\square	$(1, 0^{N-2})$	N	$(N^2 - 1)/(2N)$
$\begin{smallmatrix} \blacksquare \end{smallmatrix}$	$(0^{N-2}, 1)$	N	$(N^2 - 1)(2N)$
$\begin{smallmatrix} \blacksquare \blacksquare \end{smallmatrix}$	$(1, 0^{N-3}, 1)$	$N^2 - 1$	N
$\begin{smallmatrix} \blacksquare \blacksquare \end{smallmatrix}$	$(2, 0^{N-2})$	$N(N+1)/2$	$(N-1)(N+2)/N$
$\begin{smallmatrix} \blacksquare \\ \blacksquare \end{smallmatrix}$	$(0, 1, 0^{N-3})$	$N(N-1)/2$	$(N+1)(N-2)/N$
$\begin{smallmatrix} \blacksquare \blacksquare \blacksquare \end{smallmatrix}$	$(3, 0^{N-2})$	$N(N+1)(N+2)/6$	$3(N-1)(N+3)/(2N)$
$\begin{smallmatrix} \blacksquare \blacksquare \\ \blacksquare \end{smallmatrix}$	$(1, 1, 0^{N-3})$	$N(N^2 - 1)/3$	$3(N^2 - 3)/(2N)$
$\begin{smallmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{smallmatrix}$	$(0, 0, 1, 0^{N-4})$	$N(N-1)(N-2)/6$	$3(N+1)(N-3)/(2N)$
$\begin{smallmatrix} \blacksquare \blacksquare \blacksquare \end{smallmatrix}$	$(2, 0^{N-3}, 1)$	$N(N+2)(N-1)/2$	$(3N-1)(N+1)/(2N)$
$\begin{smallmatrix} \blacksquare \blacksquare \\ \blacksquare \end{smallmatrix}$	$(0, 1, 0^{N-4}, 1)$	$N(N-2)(N+1)/2$	$(3N+1)(N-1)/(2N)$
$\begin{smallmatrix} \blacksquare \blacksquare \blacksquare \end{smallmatrix}$	$(1, 0^{N-3}, 2)$	$N(N+2)(N-1)/2$	$(3N-1)(N+1)/(2N)$
$\begin{smallmatrix} \blacksquare \blacksquare \end{smallmatrix}$	$(1, 0^{N-4}, 1, 0)$	$N(N-2)(N+1)/2$	$(3N+1)(N-1)/(2N)$

Table 1. Invariant Casimir for low-lying representations of $SU(N)$

A.2 Replica method

We calculate the EE for a $1 + 1$ dimensional lattice with spatial lattice points L and temporal lattice points T (see fig. 3) using the formula

$$\begin{aligned}
S(A, LT) &= -\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \text{tr}_{\mathcal{H}_A} \rho_A^n = -\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \log(\text{tr}_{\mathcal{H}_A} \rho_A^n), \\
&= \lim_{n \rightarrow 1} \frac{1}{1-n} \log \text{tr}_{\mathcal{H}_A} \rho_A^n,
\end{aligned} \tag{A.13}$$

where ρ_A is the reduced density matrix $\rho_A = \text{tr}_{\mathcal{H}_{\bar{A}}} \rho$, and $\text{tr} \rho_A^n$ can be evaluated by the replica method as

$$\text{tr} \rho_A^n = \frac{Z_n(LT)}{Z_1^n}, \quad Z_1 = Z_1(LT), \tag{A.14}$$

where $Z_1(LT)$ is the unnormalized partition function of the original theory.

As before, we consider the region A and its complement \bar{A} in 1-dimension as the union of ℓ disjoint regions A_i and B_i . The whole space can be expressed as $(A_1, B_1, A_2, B_2, \dots, A_\ell, B_\ell)$.

A.3 Calculation

Let us calculate $Z_n(LT)$ using the character expansion. It is easy to see that eq. (A.2) leads to

$$\begin{aligned}
Z_n(LT) &= \int \mathcal{D}U \prod_{k=1}^n \sum_R d_R \lambda_R^{LT}(\beta) \chi_R(A_1[k] B_1[k] A_2[k] B_2[k] \cdots A_\ell[k] B_\ell[k] \\
&\quad \times C[k] B_\ell^\dagger[k] A_\ell^\dagger[k+1] \cdots B_1^\dagger[k] A_1^\dagger[k+1] D^\dagger[k])
\end{aligned} \tag{A.15}$$

$$\equiv \int_{A \cup C \cup D} \mathcal{D}U \prod_{k=1}^n Z_k(A), \tag{A.16}$$

where $A_i[k]$ and $B_i[k]$ represent the ordered products of the spatial links in the regions A_i and B_i in the k -th replica while $C[k]$ and $D[k]$ are the ordered products of the temporal links at spatial boundaries, $x = L$ and $x = 0$. LT is the number of the plaquettes in one-replica. Here $\int \mathcal{D}U$ represents integrations of all links on A, B, C, D , while $\int_{A \cup C \cup D} \mathcal{D}U$ means integrations of links on A, C, D only. Since the region A_i in k -th replica is connected to the same region in the $k + 1$ replica, $A_i[k + 1]^\dagger$ appears in the above formula, while the trace over $B_i[k]$ is implied within k -th replica. Note that $A_i[n + 1] = A_i[1]$. See fig. 3 for the $n = 3$ case.

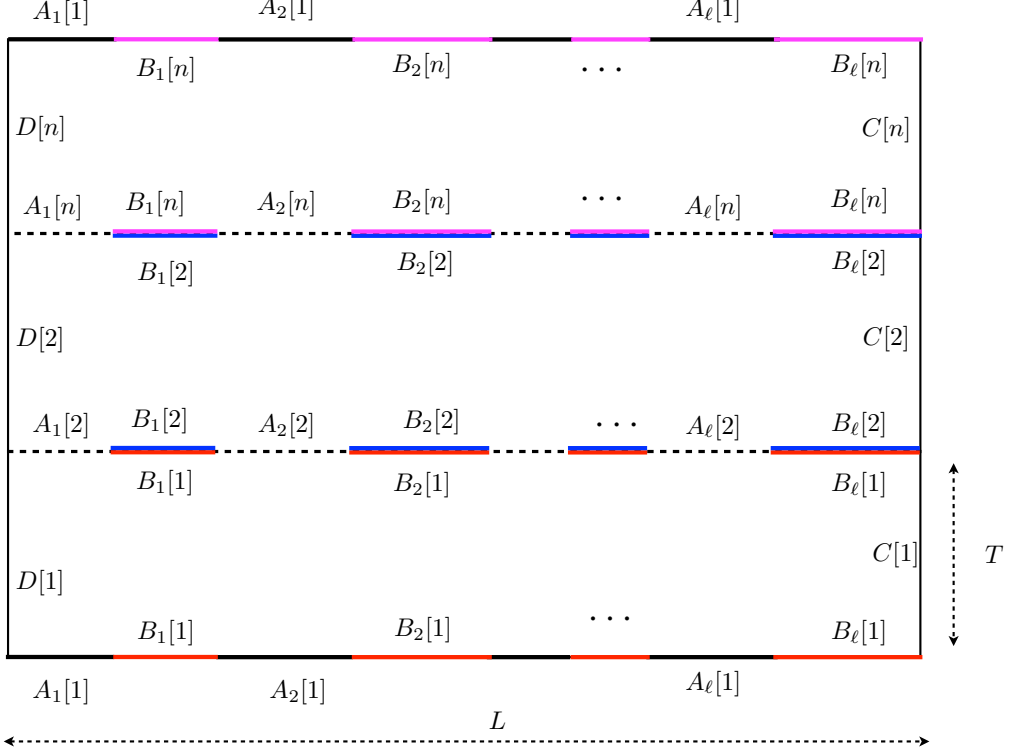


Figure 3. The replica configuration for $n = 3$.

For each k , we integrate over Ω_i of $B_i[k] = \Omega_i \hat{B}_i[k]$ using eq. (2.10) (see fig. 4), and obtain

$$Z_k(A) = \sum_R d_R^{1-\bar{\ell}} \lambda_R(\beta)^{LT} \chi_R(D[k]^\dagger A_1[k] A_1^\dagger[k+1]) \prod_{i=2}^{\ell} \chi_R(A_i[k] A_i^\dagger[k+1]) \times \chi_R(C[k]) . \quad (\text{A.17})$$

We then integrate over U_i of $A_i[k] = U_i \hat{A}_i[k]$ in $Z_n(A)$ using eq. (2.9) as

$$\int dU_i \chi_R(A_i[k-1] A_i^\dagger[k]) \chi_{R'}(A_i[k] A_i^\dagger[k+1]) = \delta_{RR'} \frac{1}{d_R} \chi_R(A_i[k-1] A_i^\dagger[k+1]) \quad (\text{A.18})$$

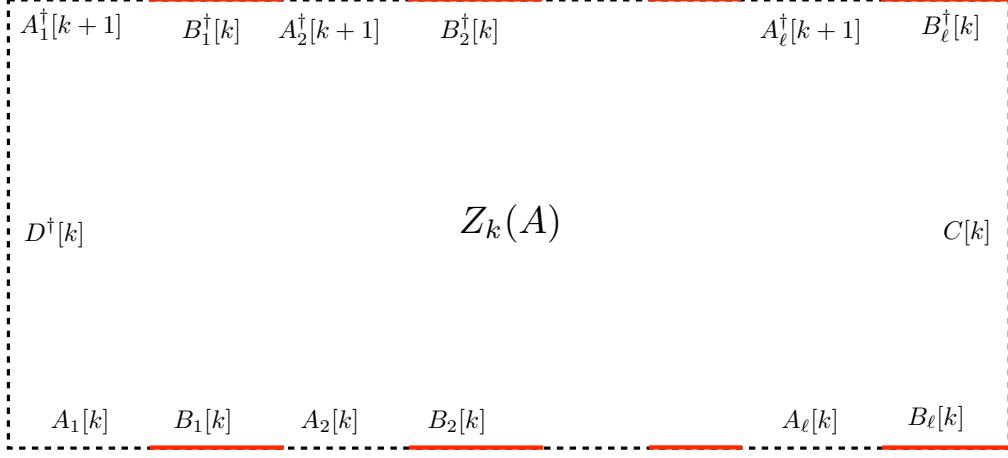


Figure 4. The configuration of $Z_k(A)$.

and

$$\begin{aligned} & \int dU_1 \chi_R(D[k-1]^\dagger A_1[k-1] A_1^\dagger[k]) \chi_{R'}(D[k]^\dagger A_1[k] A_1^\dagger[k+1]) \\ &= \delta_{RR'} \frac{1}{d_R} \chi_R(D[k]^\dagger D[k-1]^\dagger A_1[k-1] A_1^\dagger[k+1]), \end{aligned} \quad (\text{A.19})$$

which lead to

$$Z_n(A) = \sum_R \lambda_R^{nLT}(\beta) d_R^{-(n-1)(2\ell-1)} \chi_R\left(\prod_{k'=n}^1 D[k']^\dagger\right) \prod_{k=1}^n \chi_R(C[k]). \quad (\text{A.20})$$

The periodic boundary condition implies the integration of eq. (A.20) over $C = D$, which gives

$$Z_n(A) = \sum_R \lambda_R^{nLT}(\beta) d_R^{-(n-1)2\ell}. \quad (\text{A.21})$$

Using the above results, we obtain

$$\text{tr } \rho_A^n = \frac{\sum_R \lambda_R^{nLT}(\beta) d_R^{-(n-1)2\ell}}{(\sum_R \lambda_R^{LT}(\beta))^n}. \quad (\text{A.22})$$

A.4 Entanglement Entropy

Taking $n \rightarrow 1$ limit, the entanglement entropy is given by

$$S(A, LT) = \log \sum_R \lambda_R^{LT}(\beta) - \frac{\sum_R \lambda_R^{LT}(\beta) \log \lambda_R^{LT}(\beta)}{\sum_R \lambda_R^{LT}(\beta)} + 2\ell \frac{\sum_R \lambda_R^{LT}(\beta) \log d_R}{\sum_R \lambda_R^{LT}(\beta)}, \quad (\text{A.23})$$

which shows that the EE does not depend on the size of the region A but depend on the number of the boundaries of A , 2ℓ . This $S(A, LT)$ completely agrees with the EE in eq. (3.25) for the thermal state at the temperature $T_B = 1/(Ta)$ with the finite size La . At the zero temperature that $1/T \rightarrow 0$ (or the thermodynamical limit that $L \rightarrow \infty$), the EE goes to zero as

$$\lim_{1/(Ta) \rightarrow 0} S(A, LT) = 2\ell \log d_0 = 0, \quad (\text{A.24})$$

since $\lambda_0(\beta) > \lambda_{R \neq 0}(\beta)$ at $\beta < \infty$ and $d_0 = 1$. This result also agrees with the one in the main text.

Even though the replica method correctly gives the EE for the vacuum state as well as the thermal state, the operator method in the main text is much more powerful to calculate the EE for 1+1 dimensional gauge theories as it can give the EE for an arbitrary state.

Let us consider the continuum limit. Since

$$\lim_{\beta \rightarrow \infty} \frac{\lambda_R^{LT}(\beta)}{\lambda_0^{LT}(\beta)} = e^{-v_R}, \quad v_R = v C_2(R), \quad (\text{A.25})$$

where $v = g^2 l_p t_p$ with $l_p = La$ and $t_p = Ta$ is the size of the 2-dimensional space-time in unit of g^{-2} , we obtain

$$S(\ell, v) = \sum_{R \neq 0} (v_R + 2\ell \log d_R) \frac{e^{-v_R}}{f(v)} + \log f(v), \quad (\text{A.26})$$

where

$$f(v) = 1 + \sum_{R \neq 0} e^{-v_R}. \quad (\text{A.27})$$

This result shows that the EE is finite in the continuum limit and agrees with the previous result, eq. (18) of ref. [38],

$$S(A) = 2\ell v + \ln \left(\sum_R e^{-\frac{1}{2T_B N} C_2(R)} \right) - \frac{\sum_R e^{-\frac{1}{2T_B N} C_2(R)} \ln d_R^{-2\ell} e^{-\frac{1}{2T_B N} C_2(R)}}{\sum_R e^{-\frac{1}{2T_B N} C_2(R)}}, \quad (\text{A.28})$$

obtained directly in the continuum theory, with the identification that $1/(2T_B N) = g^2 l_p t_p$. Note that our result corresponds to the specific value of the UV regularization parameter v , $v = 0$, in eq. (18) of ref. [38].

A.5 Gauge fixing and gauge invariance

As discussed in the main text, the EE depends on the gauge fixing if gauge transformations at boundary points are employed.

We start with $Z_n(A)$ in eq. (A.15). We can set all spatial link variables to an unit matrix except one in each A_i or B_i , using gauge transformations at lattice points inside each region without boundary points. After this gauge fixing, $Z_n(A)$ is given by eq. (A.15)

with the replacement that $A_i[k] \rightarrow \tilde{U}_i[k]$ and $B_i[k] \rightarrow \tilde{V}_i[k]$, where $\tilde{U}_i[k]$ or $\tilde{V}_i[k]$ a (non-gauge fixed) link variable in A_i or B_i of the k -th replica (fig. 1), while $C[k] = D[k]$ are unchanged. It is now clear that we obtain the same result, eq. (A.21), after integrating out $U_i[k]$ and $V_i[k]$. The gauge invariance of $Z_n(A)$ leads to the gauge invariance of $S(A)$.

Using all gauge transformation including those on the boundaries, we can set all spatial link variables in $A \cup \bar{A}$ to an unit matrix except one. We take non-trivial link variable in A_1 for each replica and denote it as $U[k]$. We then have

$$Z_n(A) = \prod_{k=1}^n Z_k(A) \quad (\text{A.29})$$

where

$$Z_k(A) = \left\langle \sum_R d_R \lambda_R^{LT}(\beta) \chi_R(D^\dagger[k] U[k] C[k] U[k+1]) \right\rangle, \quad (\text{A.30})$$

which leads to

$$\begin{aligned} Z_n(A) &= \sum_R \lambda_R^{nLT}(\beta) d_R \left\langle \chi_R \left(\prod_{k=n}^1 D[k]^\dagger U[1] \prod_{k=1}^n C[k] U^\dagger[1] \right) \right\rangle \\ &= \sum_R \lambda_R^{nLT}(\beta) \left\langle \chi_R \left(\prod_{k=n}^1 D[k]^\dagger \right) \chi_R \left(\prod_{k=1}^n C[k] \right) \right\rangle = \sum_R \lambda_R^{nLT}(\beta). \end{aligned} \quad (\text{A.31})$$

For more general gauge fixings including some boundary points, there appear s disjoint regions \tilde{A}_i and \tilde{B}_i with $i = 1, 2, \dots, s$, each of which has at least one non-unity link. Here we can have $1 \leq s \leq \ell$. In this case, we have

$$Z_n(A, s) = \sum_R \lambda_R^{nLT}(\beta) d_R^{-(n-1)2s}. \quad (\text{A.32})$$

Considering all cases, we have

$$S(A, LT) = \log \sum_R \lambda_R^{LT}(\beta) - \frac{\sum_R \lambda_R^{LT}(\beta) \log \lambda_R^{LT}(\beta)}{\sum_R \lambda_R^{LT}(\beta)} + 2s \frac{\sum_R \lambda_R^{LT}(\beta) \log d_R}{\sum_R \lambda_R^{LT}(\beta)} \quad (\text{A.33})$$

where $s = 0, 1, 2, \dots, \ell$, for the EE after some gauge fixing.

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